

EIGENVALUES OF CURVATURE, LYAPUNOV EXPONENTS AND HARDER-NARASIMHAN FILTRATIONS

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ABSTRACT. Inspired by Katz-Mazur theorem on crystalline cohomology, also base on Eskin-Kontsevich-Zorich's numerical experiments, we conjecture that the polygon of Lyapunov spectrum lies above (or on) the Harder-Narasimhan polygon on Teichmüller curves. The conjecture is verified for almost all Teichmüller curves in low genus strata. We also discuss the connections between them and the integral of eigenvalues of the Hodge bundle curvature by using Atiyah-Bott, Forni and Möller's works.

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1. INTRODUCTION

Let \mathcal{M}_g be the moduli space of Riemann surfaces of genus g , and $\mathcal{H}_g \rightarrow \mathcal{M}_g$ the bundle of pairs (X, ω) , where $\omega \neq 0$ is a holomorphic 1-form on $X \in \mathcal{M}_g$. Denote

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$\mathcal{H}_g(m_1, \dots, m_k) \hookrightarrow \mathcal{H}_g$ the stratum of pairs (X, ω) , where $\omega (\neq 0)$ have k distinct zeros of order m_1, \dots, m_k respectively. ([KZ03])

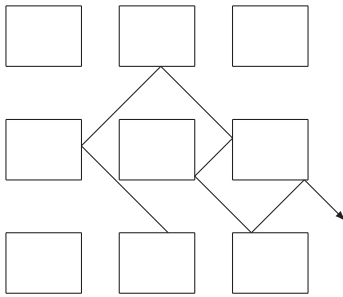
There is a natural action of $GL_2^+(\mathbb{R})$ on $\mathcal{H}_g(m_1, \dots, m_k)$, whose orbits project to complex geodesics (Teichmüller geodesic flows) in \mathcal{M}_g . The projection of an orbit is almost always dense. Teichmüller geodesic flows have strong connections with flat surfaces, billiards in polygons and interval exchange transformations. ([Zo06])

Fix an $SL_2(\mathbb{R})$ -invariant, ergodic measure μ on \mathcal{H}_g . Zorich introduce the Lyapunov exponents for the Teichmüller geodesic flow on \mathcal{H}_g

$$1 = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_g \geq 0,$$

which measure the logarithm of the growth rate of the Hodge norm of cohomology classes under the parallel transport along the geodesic flow. ([Zo94])

Ehrenfest wind-tree model for Lorenz gases [DHL11].



$$\lambda_2 = \lim_{time \rightarrow \infty} \sup \frac{\log(\text{rate of billiard trajectory escapes to infinity})}{\log(\text{time})} = \frac{2}{3}.$$

FIGURE 1. Billiard in the plane with periodic rectangular obstacles.

It is possible to evaluate Lyapunov exponents approximately through computer simulation of the corresponding dynamical system. Such experiments with Rauzy–Veech–Zorich induction (a discrete model of the Teichmüller geodesic flow) performed in [Zo96], indicated a surprising rationality of the sums $\lambda_1 + \dots + \lambda_g$ of Lyapunov exponents of the Hodge bundle with respect to Teichmüller geodesic flows on strata of Abelian and quadratic differentials [KZ97]. An explanation of this phenomenon was given by Kontsevich in [Ko97] and then developed by Forni [Fo02]: It is essentially the characteristic number of the Hodge bundle. Recently Eskin, Kontsevich and Zorich find the connection between the sum of Lyapunov exponents and Siegel–Veech constants by establishing an analytic Riemann-Roch formula [EKZ11].

Zorich conjectures the positivity of λ_g and simplicity of the spectrum of Lyapunov exponents for connected components of the strata of Abelian differentials. Forni proved the first [Fo02], Avila and Viana proved the second [AV07]. The approximate values of the Lyapunov spectrum for connected components of the strata of small genus also list in Table 6.6 [KZ97] [EKZ11].

Some Lyapunov exponents are computed such as those for invariant suborbifolds of the strata of Abelian differentials in genus two by Bainbridge [Ba07] and

for certain special Teichmüller curves: triangle groups by Bouw and Möller [BM10], Wright [Wr12b], square tiled cyclic covers [EKZ11] [FMZ11a], square tiled abelian covers by Wright [Wr12a], some wind-tree models by Delecroix, Hubert and Lelièvre [DHL11], and algebraic primitives. There is an uniform upper bound of the exponents for these examples (which essentially come from algebraic geometry) in [YZ12b].

If the stabilizer $SL(X, \omega) \subset SL_2(\mathbb{R})$ of a given form is a lattice, then the projection of its orbit gives a closed, algebraic Teichmüller curve. It is surprising that the relative canonical bundle Formula 1 on the Teichmüller curve is very simple and elegant (cf. Chen-Möller [CM11], Eskin-Kontsevich-Zorich [EKZ11]). For a Teichmüller curve, Kang Zuo and the author have introduced g -numbers:

$$1 = w_1 \geq w_2 \geq \dots \geq w_g \geq 0,$$

where w_i is obtained by normalizing the Harder-Narasimhan filtration of the Hodge bundle. We can get upper bounds of each w_i by using some filtrations of the Hodge bundle constructed from the special structure of the relative canonical bundle formula [YZ12a] [YZ12b].

Now we have λ_i measuring the stability of dynamical system and w_i measuring the stability of algebraic geometry. As an example, the numerical data of λ_i for whole stratum and of w_i for all Teichmüller curves in this stratum are listed in Table 1. Are there any relations between them?

Originally inspired by the Katz-Mazur theorem [Ma72], [Ma73] which tells us that the Hodge polygon lies above (or on) the Newton polygon of the crystalline cohomology, we make the following conjecture based on checking over all existing numerical examples:

Conjecture: The polygon of Lyapunov exponents lies above (or on) the Harder-Narasimhan polygon on Teichmüller curves.

(Warning: Different articles have different definitions for convex polygons, lies above and lies below. For the definition in this paper see Section 3 and 4.)

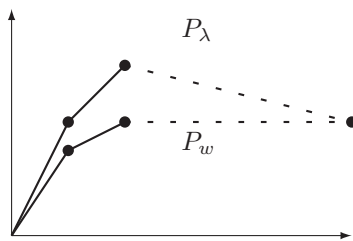


FIGURE 2. Does the Lyapunov polygon P_λ lie above (or on) the Harder-Narasimhan polygon P_w ?

TABLE 1. Genus 3 case: Approximate values of λ_i for all the strata of genus 3 [KZ97] and of w_i for all Teichmüller curves in this stratum (Theorem 3.3 and 3.4 or [YZ12a], [YZ12b]).

zeros	component	λ_2	λ_3	w_2	w_3	$\sum w_i$
(4)	hyp	0.6156	0.1844	3/5	1/5	9/5
(4)	odd	0.4179	0.1821	2/5	1/5	8/5
(3,1)		0.5202	0.2298	2/4	1/4	7/4
(2,2)	hyp	0.6883	0.3117	2/3	1/3	2
(2,2)	odd	0.4218	0.2449	1/3	1/3	5/3
(2,1,1)		0.5397	0.2936	1/2	1/3	11/6
(1,1,1,1)		0.5517	0.3411	$\leq 1/2$	$\leq 1/2$	≤ 2

The conjecture says

$$\begin{cases} \sum_{j=1}^i \lambda_j \geq \sum_{j=1}^i w_j \text{ for } i = 1, \dots, g-1; \\ \sum_{j=1}^g \lambda_j = \sum_{j=1}^g w_j. \end{cases}$$

Or equivalently to say : $\sum_{j=i}^g \lambda_j \leq \sum_{j=i}^g w_j$, for $i = 2, \dots, g$; and $\sum_{j=1}^g \lambda_j = \sum_{j=1}^g w_j$.

The conjecture was first announced by the author in the Oberwolfach conference [Yu14]. After that we know that this conjecture is analogous to the work of Atiyah-Bott on Hermitian-Yang-Mills metric [AB82]. This analogy was also noticed separately by Möller. Here we state the result of Atiyah-Bott in a different form so that it is easy to see this analogy. Let $\varepsilon_j (1 \leq j \leq g)$ be the normalized integral of the j -th eigenvalue of the Hodge bundle curvature, see Forni [Fo02]. It follows that

$$1 = \varepsilon_1 \geq \varepsilon_2 \geq \dots \geq \varepsilon_g \geq 0.$$

It follows [AB82] that

$$\begin{cases} \sum_{j=1}^i \varepsilon_j \geq \sum_{j=1}^i w_j \text{ for } i = 1, \dots, g-1; \\ \sum_{j=1}^g \varepsilon_j = \sum_{j=1}^g w_j. \end{cases}$$

Since we have got upper bounds for each w_i [YZ12b], the above formula provides some information about ε_i . We remark here that some λ_i and ε_i are also related, see [Fo02] or Section 4.2.

In Section 2 we review the definition of Teichmüller curves, its relative canonical bundle formula and the natural filtrations of the Hodge bundle. Section 3 summarized facts about slope filtrations, especially Lyapunov exponents, Harder-Narasimhan filtrations and the integral of eigenvalues of the curvature. In Section 4 we concentrate on convexity appeared in geometry and arithmetic. The Section 4.1 is about polygons of eigenvalue spectrum and Harder-Narasimhan polygons, the

Section 4.2 is about polygons of eigenvalue spectrum and Lyapunov polygons and the third one is about Hodge polygons and Newton polygons.

In Section 5 firstly we describe our conjectures and show that the main conjecture is correct for almost all Teichmüller curves in low genus strata, then give some applications and prove a simple corollary $w_i = 0$ implies $\lambda_i = 0$ by using Higgs bundles, at last we reprove a conjecture of Kontsevich-Zorich. We give the numerical evidence for the main conjecture in Appendix.

2. TEICHMÜLLER CURVES

Teichmüller geodesic flows have close connections with flat surfaces, billiards in polygons and interval exchange transformations. Zorich gives a survey [Zo06] about many important ideas of this field. There is also a survey [Mö12] by Möller about Teichmüller curves mainly from the view point of algebraic geometry.

Denote by $\mathcal{H}_g(m_1, \dots, m_k)$ the stratum parameterized by (X, ω) where X is a curve of genus g and ω is an Abelian differential (i.e. a holomorphic one-form) on X that have k distinct zeros of order m_1, \dots, m_k . Let $\overline{\mathcal{H}}_g(m_1, \dots, m_k)$ be the Deligne-Mumford compactification of $\mathcal{H}_g(m_1, \dots, m_k)$. Denote by $\mathcal{H}_g^{hyp}(m_1, \dots, m_k)$ (resp. odd, resp. even) the hyperelliptic (resp. odd theta character, resp. even theta character) connected component. ([KZ03])

Let $\mathcal{Q}(d_1, \dots, d_n)$ be the stratum parameterizing (Y, q) where X is a curve of genus g and q is a meromorphic quadratic differential with at most simple zeros on X that have k distinct zeros of order d_1, \dots, d_n respectively. If the quadratic differential is not a global square of a one-form, there is a canonical double covering $\pi: X \rightarrow Y$ such that $\pi^*q = \omega^2$. This covering is ramified precisely at the zeros of odd order of q and at the poles. It gives a map

$$\phi: \mathcal{Q}(d_1, \dots, d_n) \rightarrow \mathcal{H}_g(m_1, \dots, m_k).$$

A singularity of order d_i of q gives rise to two zeros of degree $m = d_i/2$ when d_i is even, single zero of degree $m = d_i + 1$ when d_i is odd. Especially, the hyperelliptic locus in a stratum $\mathcal{H}_g(m_1, \dots, m_k)$ induces from a stratum $\mathcal{Q}(d_1, \dots, d_n)$ satisfying $d_1 + \dots + d_n = -4$. ([EKZ11])

There is a natural action of $GL_2^+(\mathbb{R})$ on $\mathcal{H}_g(m_1, \dots, m_k)$, whose orbits project to complex geodesics (**Teichmüller geodesic flows**¹) in \mathcal{M}_g . The projection of an orbit is almost always dense. If the stabilizer $SL(X, \omega) \subset SL_2(\mathbb{R})$ of given form is a lattice, however, then the projection of its orbit gives a closed, algebraic **Teichmüller curve** ([Mc06])

$$\rho: V = \mathbb{H}/SL(X, \omega) \rightarrow \mathcal{M}_g.$$

The Teichmüller curve V is an algebraic curve in \mathcal{M}_g that is totally geodesic with respect to the Teichmüller metric. After suitable base change and compactification, we can get a universal family $f: S \rightarrow C$, which is a relatively minimal semi-stable model with disjoint sections D_1, \dots, D_k ; restrictions to each fiber X , $D_i|_X$ is a zero of ω . ([CM11, p.11][Mö06])

Let $\mathcal{L} \subset f_*\omega_{S/C}$ be the line bundle whose fiber over the point corresponding to X is $\mathbb{C}\omega$, the generating differential of Teichmüller curves; it is also known as the "maximal Higgs" line bundle [Mö06]. Let $\Delta \subset C$ be the set of points with

¹The closure are algebraic varieties by the works of Eskin-Mirzakhani [EM13], Eskin-Mirzakhani-Mohammadi [EMM13] and Filip [Fi13a] [Fi13b].

singular fibers, then the property of being "maximal Higgs" says by definition that $\mathcal{L} \cong \mathcal{L}^{-1} \otimes \omega_C(\log \Delta)$ [VZ04], and so

$$\chi := 2\deg \mathcal{L} = 2g(C) - 2 + |\Delta|.$$

The relative canonical bundle formula is very simple ([CM11, p.18-19] [EKZ11, p.33]):

$$(1) \quad \omega_{S/C} \simeq f^* \mathcal{L} \otimes \mathcal{O}_S(\sum_{i=1}^k m_i D_i).$$

By the adjunction formula we get

$$D_i^2 = -\omega_{S/C} D_i = -m_i D_i^2 - \deg \mathcal{L},$$

and the self-intersection number of D_i is

$$D_i^2 = -\frac{1}{m_i + 1} \frac{\chi}{2}.$$

Let $h^0(\mathcal{V})$ be the dimension of $H^0(X, \mathcal{V})$. If $0 \leq d_i \leq m_i$, then from the exact sequence

$$0 \rightarrow f_* \mathcal{O}(d_1 D_1 + \dots + d_k D_k) \rightarrow f_* \mathcal{O}(m_1 D_1 + \dots + m_k D_k) = f_* \omega_{S/C} \otimes \mathcal{L}^{-1}$$

and the fact that all sub-sheaves of a locally free sheaf on a curve are locally free, we deduce that

$$f_* \mathcal{O}(d_1 D_1 + \dots + d_k D_k) \text{ is a sub-vector bundle of rank } h^0(d_1 p_1 + \dots + d_k p_k),$$

where p_i is the intersection point of the section D_i and a generic fiber F . By varying d_i , we have constructed many filtrations of the Hodge bundle by using those vector bundles in [YZ12a].

The fundamental exact sequence

$$0 \rightarrow f_* \mathcal{O}(\sum (d_i - a_i) D_i) \rightarrow f_* \mathcal{O}(\sum d_i D_i) \rightarrow f_* \mathcal{O}_{\sum a_i D_i}(\sum d_i D_i) \xrightarrow{\delta}$$

$$R^1 f_* \mathcal{O}(\sum (d_i - a_i) D_i) \rightarrow R^1 f_* \mathcal{O}(\sum d_i D_i) \rightarrow 0$$

implies many good properties for these filtrations. In particular we have

Lemma 2.1 ([YZ12a]). *The Harder-Narasimhan filtration of $f_* \mathcal{O}_{aD}(dD)$ is*

$$0 \subset f_* \mathcal{O}_D((d - a + 1)D) \subset \dots \subset f_* \mathcal{O}_{(a-1)D}((d - 1)D) \subset f_* \mathcal{O}_{aD}(dD)$$

and the direct sum of the graded quotient of this filtration is

$$\text{grad}(HN(f_* \mathcal{O}_{aD}(dD))) = \bigoplus_{i=0}^{a-1} \mathcal{O}_D((d - i)D).$$

By using those filtrations, we obtained Theorem 3.3 and Theorem 3.4 in [YZ12a] and [YZ12b]. They describe the Harder-Narasimhan polygon of Hodge bundles on Teichmüller curves.

3. SLOPE FILTRATIONS

André [An08] surveys many slope filtrations occur in algebraic and analytic geometry, in asymptotic analysis, in ramification theory, in p -adic theories, in geometry of numbers. Five basic examples are the Harder-Narasimhan filtration of vector bundles over a smooth projective curve, the Dieudonné-Manin filtration of F -isocrystals over the p -adic point, the Turrittin-Levelt filtration of formal differential modules, the Hasse-Arf filtration of finite Galois representations of local fields, and the Grayson-Stuhler filtration of Euclidean lattices. Despite the variety of their origins, these filtrations share a lot of similar features.

For some object N , there is a unique descending **slope filtration**

$$0 \subset F^{\geq \lambda_1} N \subset \dots \subset F^{\geq \lambda_r} N = N$$

for which $\lambda_1 > \dots > \lambda_r$, and $\text{gr}^{\lambda_i} N = F^{\geq \lambda_i} N / F^{> \lambda_i} N$ is isoclinic of slope λ_i . Denote $\text{rk}(\text{gr}^{\lambda_i} N)$ by n_i , and $n = \sum n_i$. We shall call the sequence of pairs $(n_i, n_i \lambda_i)$, $i = 1, \dots, r$ the type of N . It is sometimes convenient to describe the type equivalently by the single n -vector μ whose components are the ratios λ_i each represented n_i times and arranged in decreasing order. Thus

$$\mu = (\mu_1, \dots, \mu_n)$$

with $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n$, where the first n_i are equal to λ_i , the next n_2 are equal to λ_2 and so on.

We introduce a partial ordering on the vectors μ that parameterize our types. This partial ordering can be described in several equivalent ways. First we associate with the type μ the **convex polygon** P_μ with vertices

$$(0, 0), (1, \mu_1), (2, \mu_1 + \mu_2), \dots, (n, \mu_1 + \dots + \mu_n).$$

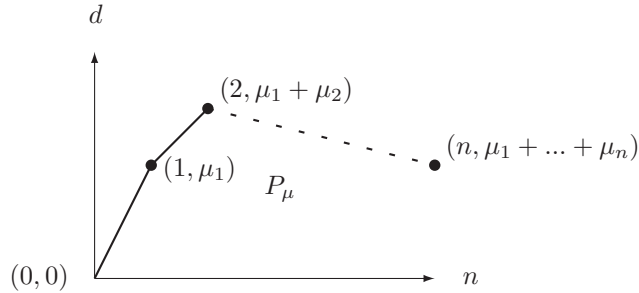


FIGURE 3. Convex polygon P_μ .

Note that the convexity of P_μ is equivalent to monotonicity of λ_i .

3.1. Eigenvalues of curvature: ε . Forni introduces the eigenvalues of curvature to study Lyapunov exponents in [Fo02]. Here we follow [FMZ11] whose setup is similar to this paper.

Let

$$f: \overline{\mathcal{M}}_{g,1} \rightarrow \overline{\mathcal{M}}_g$$

be natural forgetful map from the compactification moduli space of $(X, p \in X)$ to the compactification moduli space of Riemann surface X of genus g .

For the weight one \mathbb{Q} -VHS

$$(R^1 f_* \mathbb{Q}, H^{1,0} = f_* \omega_{\overline{\mathcal{M}}_{g,1}/\overline{\mathcal{M}}_g} \subset H = (R^1 f_* \mathbb{Q} \otimes_{\mathbb{Q}} \mathcal{O}_{\mathcal{M}_g})_{ext})$$

The flat Gauss-Manin connection ∇ composed with the inclusion and projection give a map

$$A^{1,0}: H^{1,0} \rightarrow H \rightarrow H \otimes \Omega_{\overline{\mathcal{M}}_g}(\log(\overline{\mathcal{M}}_g \setminus \mathcal{M}_g)) \rightarrow (H/H^{1,0}) \otimes \Omega_{\overline{\mathcal{M}}_g}(\log(\overline{\mathcal{M}}_g \setminus \mathcal{M}_g)),$$

which is $\mathcal{O}_{\overline{\mathcal{M}}_g}$ -linear.

The $A^{1,0}$ is the second fundamental form, which is also known as the Kodaira-Spencer map. When restricted to a curve $C \subset \overline{\mathcal{M}}_g$, $A^{1,0} \wedge A^{1,0} = 0$. It is also known as the Higgs field which is discussed in Section 5.2.

Denote by $\Theta_H, \Theta_{H^{1,0}}, \Theta_{H^{0,1}}$ the curvature tensor of the metric connections of the holomorphic Hermitian bundle $H, H^{1,0}, H^{0,1}$. By Cartan's structure equation, we conclude that

$$\Theta_H = \begin{bmatrix} \Theta_{H^{1,0}} - \overline{A^{1,0}}^T \wedge A^{1,0} & * \\ * & \Theta_{H^{0,1}} - A^{1,0} \wedge \overline{A^{1,0}}^T \end{bmatrix}$$

It follows that

$$\Theta_{H^{1,0}} = \Theta_H|_{H^{1,0}} + \overline{A^{1,0}}^T \wedge A^{1,0}.$$

Note that Θ_H is the curvature of the Gauss-Manin connection, which is flat. So Θ_H is null, and the curvature can be written as:

$$\Theta_{H^{1,0}} = \overline{A^{1,0}}^T \wedge A^{1,0}.$$

We work with the pullbacks of the vector bundles $H, H^{1,0}, H^{0,1}$ to the moduli spaces \mathcal{H}_g or \mathcal{Q}_g of Abelian (correspondingly quadratic) differentials with respect to the natural projections $\rho: \mathcal{H}_g \rightarrow \mathcal{M}_g$ (correspondingly $\varrho: \mathcal{Q}_g \rightarrow \mathcal{M}_g$). For any (X, q) at the tangent vector $v = q$ under the identification between the bundle of quadratic differentials and the tangent bundle of the moduli space of Riemann surfaces (through Beltrami differentials [FMZ11, p.8])

$$A_q: H^{1,0}(X) \rightarrow H^{0,1}(X).$$

For any Abelian differential ω , let $A_\omega := A_q$ be the complex-linear map corresponding to the quadratic differential $q = \omega^2$.

For any $\alpha, \beta \in H^{1,0}(X)$, we define:

$$B_\omega(\alpha, \beta) := \frac{i}{2} \int_X \frac{\alpha \beta}{\omega} \overline{\omega}.$$

The complex symmetry bilinear form depends continuously, in fact (real) analytically, on the Abelian differential ω . The second fundamental form A_ω can be written in terms of the complex symmetry bilinear form B_ω [Fo02][FMZ11, Lemma 2.1]:

$$(A_\omega(\alpha), \overline{\beta}) = -B_\omega(\alpha, \beta).$$

It is related to the derivative of the period matrix along the Teichmüller geodesic flow.

For any Abelian differential ω , let H_ω be the *negative* of the Hermitian curvature form Θ_ω on $H^{1,0}(X)$. Let B be the matrix of the bilinear form B_ω on $H^{1,0}(X)$ with respect to the basis $\Omega := \{\omega_1, \dots, \omega_g\}$, that is:

$$B_{jk} := \frac{i}{2} \int_S \frac{\omega_j \omega_k}{\omega} \overline{\omega}.$$

The Hermitian form H_ω is positive-semidefinite and its matrix H with respect to any Hodge-orthonormal basis Ω can be written as follows [Fo02][FMZ11]:

$$H = B \cdot \overline{B}^T.$$

Let $EV(H_\omega)$ and $EV(B_\omega)$ denote the set of eigenvalues of the forms H_ω and B_ω respectively. The following identity holds:

$$EV(H_\omega) = \{|\lambda|^2 | \lambda \in EV(B_\omega)\}.$$

For every Abelian differential ω , the eigenvalues of the positive semidefinite form H_ω on $H^{1,0}(X)$ will be denoted as follows [Fo02][FMZ11, p.16]:

$$1 = \Lambda_1(\omega) > \Lambda_2(\omega) \geq \dots \geq \Lambda_g(\omega) \geq 0.$$

In particular all of the above eigenvalues give well defined continuous, non-negative, bounded function on the moduli space of all (normalized) abelian differentials.

For a Teichmüller curve C , there is a volume form $d\sigma$ which satisfies

$$\frac{i}{2\pi} \Theta_{H^{1,0}} = H_\omega d\sigma$$

The volume form is also equal to $dg_{hyp}(\omega)/\pi$ which appears in [EKZ11, p.32]. And moreover we have

$$\int_C \Lambda_1(\omega) d\sigma = \int_C d\sigma = \frac{\chi}{2}.$$

Following Forni [Fo02], we define the integral

$$(2) \quad \varepsilon_j = \int_C \Lambda_j(\omega) d\sigma / \frac{\chi}{2}.$$

This is g -number

$$1 = \varepsilon_1 \geq \dots \geq \varepsilon_g \geq 0.$$

We denote the eigenvalue type of C by the g -vector

$$\varepsilon(C) = (\varepsilon_1, \dots, \varepsilon_g).$$

3.2. Lyapunov exponents: λ . Zorich introduces the Lyapunov exponents to study Teichmüller geodesic flows [Zo94]. The geometry meaning of Lyapunov exponents is clearly explained in [Zo06, section 4].

A motivating example called Ehrenfest wind-tree model for Lorenz gases appears in the work of Delecroix, Hubert and Lelièvre [DHL11]. Consider a billiard on the plane with \mathbb{Z}^2 -periodic rectangular obstacles as in Figure 1.

It is shown that for almost all parameters (a, b) of the obstacle (i.e., lengths $a, b \in (0, 1)$ of the sides of the rectangular obstacles), for almost all initial direction θ , and for any starting point x the billiard trajectory escapes to infinity with a rate $t^{2/3}$.

The number " $\frac{2}{3}$ " here is the Lyapunov exponent of a certain renormalizing dynamical system associated to the initial one.

$$\lambda_2 = \limsup_{t \rightarrow \infty} \frac{\log(\text{distance between } x \text{ and } \phi_t^\theta(x))}{\log t} = \frac{2}{3}.$$

Now we give a general definition of Lyapunov exponents. Fix an $SL_2(\mathbb{R})$ -invariant, ergodic measure μ on \mathcal{H}_g . Let V be the restriction of the real Hodge bundle (i.e. the bundle with fibers $H^1(X, \mathbb{R})$) to the support M of μ . Let S_t be

the lift of the geodesic flow to V via the Gauss-Manin connection. Then *Oseledec's multiplicative ergodic Theorem* guarantees the existence of a filtration

$$0 \subset V_{\lambda_g} \subset \dots \subset V_{\lambda_1} = V$$

by measurable vector subbundles with the property that, for almost all $m \in M$ and all $v \in V_m \setminus \{0\}$ one has

$$\|S_t(v)\| = \exp(\lambda_i t + o(t)),$$

where i is the maximal index such that v is in the fiber of V_i over m (i.e. $v \in (V_i)_m$). The numbers λ_i for $i = 1, \dots, k \leq \text{rank}(V)$ are called the **Lyapunov exponents** of S_t ($S_t(v)$ is called **Kontsevich-Zorich cocycle**). Since V is symplectic, the spectrum is symmetric in the sense that $\lambda_{g+k} = -\lambda_{g-k+1}$. Moreover, from elementary geometric arguments it follows that one always has $\lambda_1 = 1$. Thus, the Lyapunov spectrum is defined by the non-negative Lyapunov exponents

$$1 = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_g \geq 0.$$

We will apply Oseledec's theorem in two instances. The first are Masur-Veech measures μ_{gen} with support equal to the whole hypersurface of flat surfaces of area one in a connected component. When talking about Lyapunov exponents for Teichmüller curves C we take μ to be the measure on the unit tangent bundle T^1C to a Teichmüller curve that stems from the Poincaré metric g_{hyp} on \mathbb{H} with scalar curvature -4 . In both cases, the integrability condition of Oseledec's theorem has to be verified. ([Mö12, p.38])

We denote the Lyapunov type of C by the g -vector

$$\lambda(C) = (\lambda_1, \dots, \lambda_g).$$

The bridge between the "dynamical" definition of Lyapunov exponents and the "algebraic" method applied in the sequel is given by the following result. It is first formulated by Kontsevich [Ko97] (in a slightly different form) and then proved rigorously by Forni [Fo02].

Theorem 3.1 ([KZ97] [Fo02][BM10]). *If the VHS over the Teichmüller curve C contains a sub-VHS \mathbb{W} of rank $2k$, then the sum of the k corresponding non-negative Lyapunov exponents equals*

$$\sum_{i=1}^k \lambda_i^{\mathbb{W}} = \frac{2\deg \mathbb{W}^{(1,0)}}{2g(C) - 2 + |\Delta|},$$

where $\mathbb{W}^{(1,0)}$ is the $(1,0)$ -part of the Hodge filtration of the vector bundle associated with \mathbb{W} . In particular, we have

$$\sum_{i=1}^g \lambda_i = \sum_{i=1}^g \varepsilon_i = \frac{2\deg f_* \omega_{S/C}}{2g(C) - 2 + |\Delta|}.$$

Eskin, Kontsevich and Zorich have got an analytic Riemann-Roch formula to compute the sum of Lyapunov exponents for Teichmüller geodesic flows :

Theorem 3.2 ([EKZ11, Theorem 1]). *Let \mathcal{M}_1 be any closed connected regular $SL(2, R)$ -invariant suborbifold of some stratum $\mathcal{M}_g(m_1, \dots, m_n)$ of abelian differentials, where $m_1 + \dots, m_n = 2g - 2$. The top g Lyapunov exponents of the Hodge*

bundle over \mathcal{M}_1 along the Teichmüller flow satisfy the following relation:

$$\sum_{i=1}^g \lambda_i = \frac{1}{12} \sum_{i=1}^k \frac{m_i(m_i + 2)}{m_i + 1} + \frac{\pi^2}{3} c_{area}(\mathcal{M}_1),$$

where $c_{area}(\mathcal{M}_1)$ is the area Siegel-Veech constant corresponding to the regular sub-orbifold \mathcal{M}_1 . The leading Lyapunov exponent λ_1 is equal to one.

3.3. Harder-Narasimhan filtrations: *w.* We refer the readers to [HN75] [HL97] for details about the Harder-Narasimhan filtration.

For a smooth curve C and a holomorphic vector bundle V over C . We recall now the precise definition of stability. It is more consentient to introduce first the normalized Chern class or "slope" $\mu(V) = \deg(V)/\text{rk}(V)$. Then a holomorphic bundle V is stable if for every proper holomorphic sub-bundle W of V , we have $\mu(W) < \mu(V)$. Semi-stable is defined similarly but we allow now the weak inequality $\mu(W) \leq \mu(V)$.

Harder and Narasimhan show that every holomorphic bundle V has a canonical filtration

$$0 = HN_0(V) \subset HN_1(V) \subset \dots \subset HN_r(V) = V,$$

with the graded quotients $\text{gr}_i^{HN} = HN_i(V)/HN_{i-1}(V)$ is semi-stable and

$$\mu(\text{gr}_1^{HN}) > \mu(\text{gr}_2^{HN}) > \dots > \mu(\text{gr}_r^{HN}).$$

If gr_i^{HN} has rank n_i and Chern class k_i so that $n = \sum n_i, k = \sum k_i$ we shall call the sequence of pairs $(n_i, k_i), i = 1, \dots, r$ the type of V . As before, it is convenient to describe the type equivalently by the single n -vector $\mu(V)$ whose components are the ratios k_i/n_i each represented n_i times and arranged in decreasing order

$$\mu(V) = (\mu_1, \dots, \mu_n).$$

For a Teichmüller curve C , it is convenient to set $w_i = \mu_i(f_*\omega_{S/C})/(\chi/2)$ and denote the Harder-Narasimhan type of C by the g -vector

$$(3) \quad w(C) = (w_1, \dots, w_g).$$

Then the Harder-Narasimhan type of a Teichmüller curve is given by the following two theorems:

Theorem 3.3 ([YZ12a]). *Let C be a Teichmüller curve in the hyperelliptic locus of some stratum $\overline{\mathcal{H}}_g(m_1, \dots, m_k)$, and denote by (d_1, \dots, d_n) the orders of singularities of underlying quadratic differentials. Then w_i for C is the i -th largest number in the following set*

$$1, \quad \left\{ 1 - \frac{2k}{d_j + 2} \right\}_{\forall d_j, 0 < 2k \leq d_j + 1}$$

For a Teichmüller curve lying in some low genus non varying strata, the w_i 's are computed in Table 1, 6 and 6.

It can be used to give a complete proof of a Kontsevich-Zorich conjecture in [CM11] [YZ12a].

Theorem 3.4 ([YZ12b]). *For a Teichmüller curve which lies in $\mathcal{H}_g(m_1, \dots, m_k)$, we have inequalities:*

$$1 - a_{H_i}(P) \geq w_i$$

here a_n is the n -th smallest number in $\{\frac{l}{m_j+1} | 1 \leq l \leq m_j, 1 \leq i \leq k\}$, P is the special permutation and $H_i(P) \geq 2i - 2$.

A simple corollary of this theorem is

Corollary 3.5 ([YZ12b]). *For a Teichmüller curve which lies in $\mathcal{H}_g(m_1, \dots, m_k)$, we have inequalities:*

$$\sum_{i=1}^g \lambda_i = \sum_{i=1}^g \varepsilon_i = \sum_{i=1}^g w_i \leq \frac{g+1}{2}.$$

4. CONVEXITY

In [AB82, section 12], Atiyah and Bott discussed the convexity of polygons and the relation with Hermitian matrices. Shatz defines the partial ordering by

$$\lambda \geq \mu \text{ if } P_\lambda \text{ is above } P_\mu.$$

If we consider P_μ as the graph of a concave function p_μ , then p_μ is defined on the integers by

$$p_\mu(i) = \sum_{j \leq i} \mu_j$$

and interpolates linearly between integers. Here the μ_j are the components of our n -vector μ .

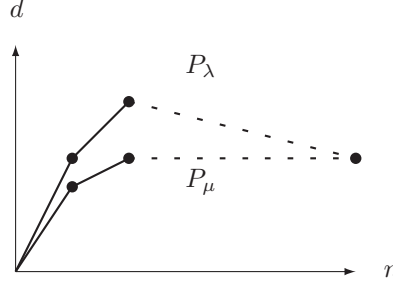


FIGURE 4. The P_λ lies above (or on) the P_μ .

Hence, for our vector notation, it translates in to the following partial ordering:

$$\lambda \geq \mu \Leftrightarrow \begin{cases} \sum_{j=1}^i \lambda_j \geq \sum_{j=1}^i \mu_j \text{ for } i = 1, \dots, n-1; \\ \sum_{j=1}^n \lambda_j = \sum_{j=1}^n \mu_j \end{cases}.$$

This partial ordering on vectors in R^n is well known in various contexts.

This partial ordering occurs in Horn [Ho54] where it is shown to be equivalent to either of the following properties

$$(4) \quad \sum_j f(\mu_i) \leq \sum_j f(\lambda_j) \text{ for every convex function } f : R \rightarrow R;$$

$\mu = P\lambda$ where $\lambda, \mu \in R^n$ and P is a doubly stochastic matrix.

We recall that a real square matrix is *stochastic* if $p_{ij} \geq 0$ and $\sum_j p_{ij} = 1$ for all

i. If in addition the transposed matrix is also stochastic then P is called *doubly stochastic*. A theorem of Birkhoff identifies doubly stochastic matrices in terms of permutation matrices, namely

The doubly stochastic $n \times n$ matrices are the convex hull of the permutation matrices.

Now the equivalence relation can be replace by

$$\widehat{\Sigma_n \mu} \subset \widehat{\Sigma_n \lambda}$$

where $\Sigma_n x$ denotes the orbit of any $x \in R^n$ under the permutation group Σ_n , and \hat{C} denotes the convex hull of the set $C \subset R^n$.

Schur showed that if $\mu_j (j = 1, \dots, n)$ are the diagonal elements of a Hermitian matrix whose eigenvalues are λ_j , then $\mu \leq \lambda$. We give the proof for the largest eigenvalue, the proof is similar for general cases.

Lemma 4.1 (Schur). *For a Hermitian matrix $H = [h_{ij}]$, let $\lambda_1 \geq \dots \geq \lambda_n$ be its eigenvalues, then*

$$\lambda_1 \geq h_{ii}.$$

Proof. Let $U = [u_{ij}]$ be a unitary matrix such that

$$H = U \text{diag}[\lambda_1, \dots, \lambda_n] \bar{U}^T.$$

Because $\sum_j u_{ij} \bar{u}_{ij} = 1$, we have

$$\lambda_1 = \lambda_1 \left(\sum_j u_{ij} \bar{u}_{ij} \right) \geq \sum_j \lambda_j u_{ij} \bar{u}_{ij} = h_{ii}.$$

□

Horn proved the converse so that another equivalent of $\mu \leq \lambda$ is

The λ_j are the eigenvalues of a Hermitian matrix with diagonal elements μ_j .

For a general compact Lie group G , the role of the Hermitian (or rather skew-Hermitian) matrices is played now by the Lie algebra \mathfrak{g} of G . The diagonal matrices are replaced by the Lie algebra \mathfrak{t} of a maximal torus T of G and Σ_n becomes the Weyl group W . Writing a set of λ_j in decreasing order corresponds to picking a (closed) positive Weyl chamber C in \mathfrak{t} : this is a fundamental domain for the action of W . ([AB82])

We also need the following linear algebra fact:

Lemma 4.2. *For a complex symmetric matrix $B = [b_{ij}]$, $H = B \bar{B}^T = [h_{ij}]$, $\alpha = [a_1, \dots, a_n]$, $\alpha \bar{\alpha}^T = 1$. we have*

$$\alpha H \bar{\alpha}^T \geq |\alpha B \alpha^T|^2.$$

Proof. There is a decomposition for any complex symmetric matrix

$$B = U \text{diag}[\lambda_1, \dots, \lambda_n] U^T,$$

where $U = [u_{ij}]$ is unitary, $\sum_i u_{ji} \bar{u}_{ji} = 1$, $\alpha = [a_1, \dots, a_n]$, $\sum_i a_i \bar{a}_i = 1$. Because

$$H = B \bar{B}^T = U \text{diag}[|\lambda_1|^2, \dots, |\lambda_n|^2] \bar{U}^T,$$

we only need to show that

$$(\alpha U) \text{diag}(|\lambda_1|^2, \dots, |\lambda_n|^2) (\overline{\alpha U})^T \geq |(\alpha U) \text{diag}(\lambda_1, \dots, \lambda_n) (\alpha U)^T|^2.$$

Let $\beta = \alpha U = [b_1, \dots, b_n]$, then $\beta \bar{\beta}^T = 1$, we need to show that

$$\beta \text{diag}(|\lambda_1|^2, \dots, |\lambda_n|^2) \bar{\beta}^T \geq |\beta \text{diag}(\lambda_1, \dots, \lambda_n) \beta^T|^2.$$

This is

$$\sum |\lambda_i|^2 |b_i|^2 \geq |\sum \lambda_i b_i^2|^2.$$

By Cauchy inequality and $\sum |b_i|^2 = 1$, we know the inequality is right

$$(\sum |\lambda_i|^2 |b_i|^2) (\sum |b_i|^2) \geq (\sum |\lambda_i| |b_i|^2)^2 \geq |\sum \lambda_i b_i^2|^2.$$

□

4.1. $\varepsilon \geq w$. There is a relation between integral of eigenvalue spectrum of the curvature and the Harder-Narasimhan filtration. The following theorem is essentially contained in Atiyah-Bott[AB82]:

Theorem 4.3. [AB82, p.573-575] *Let E be a Hermitian vector bundle of rank n on a Riemann surface M with a volume form $d\sigma$, and*

$$\Lambda_1(E) \geq \dots \geq \Lambda_n(E),$$

*are the eigenvalues of $\frac{i}{2\pi} * \Theta(E)$. For $1 \leq k \leq n$, we have*

$$\sum_{j=1}^k \int_M \Lambda_j(E) d\sigma \geq \sum_{j=1}^k \mu_j(E).$$

It is equality when $k = n$.

Proof. We shall begin by proving in the simple case when

$$\mu_1 = \mu_2 = \dots = \mu_r > \mu_{r+1} = \dots = \mu_n.$$

So that the Harder-Narasimhan filtration of the bundle E has just two steps. We have an exact sequence of vector bundles

$$0 \rightarrow D_1 \rightarrow E \rightarrow D_2 \rightarrow 0$$

where D_j has rank m_j Chern class k_j ($j = 1, 2$) so that $\mu_1 = k_1/m_1$ and $\mu_n = k_2/m_2$. For convenience we shall use the notation $\mu^j = k_j/m_j$ ($j = 1, 2$). For the connection defined by the holomorphic structure and natural Hermitian metric. The curvature $\Theta(E)$ can then be written as the form

$$\Theta(E) = \begin{bmatrix} F_1 - \eta \wedge \eta^* & d\eta \\ -d\eta^* & F_2 - \eta^* \wedge \eta \end{bmatrix},$$

where F_j is the curvature of the metric connection of D_j , $\eta \in \Omega^{0,1}(M, \text{Hom}(D_2, D_1))$, η^* is its transposed conjugate and $d\eta$ is the covariant differential. Now let f_j, α_j be scalar $m_j \times m_j$ matrices such that

$$\text{trace } f_j = \text{trace } * F_j$$

$$\text{trace } \alpha_1 = \text{trace } * (\eta \wedge \eta^*) = -\text{trace } * (\eta^* \wedge \eta) = -\text{trace } \alpha_2.$$

We know that $\frac{i}{2\pi} * \Theta(E)$ is a Hermitian matrix. By the equivalence condition 4 of convexity, some elementary inequalities concerning convex invariant function ϕ show that

$$\phi(*\Theta(E)) \geq \phi \begin{bmatrix} f_1 - \alpha_1 & 0 \\ 0 & f_2 - \alpha_2 \end{bmatrix}.$$

In particular by Lemma 4.1 it implies

$$\Lambda_1(E) \geq \frac{i}{2\pi} \frac{\text{trace}(f_1 - \alpha_1)}{m_1}.$$

But the Chern class k_j of D_j is given by

$$k_j = \frac{i}{2\pi} \int_M \text{trace } f_j d\sigma.$$

Since f_j is scalar matrix this means that $\int_M \text{trace } f_j$ is scalar matrix whose diagonal entries are $-2\pi i k_j / m_j = -2\pi i \mu^j$. Also from (since $\eta \in \Omega^{0,1}$) it follows that $-i \text{trace } \alpha_1$ is non-negative and so

$$\int_M \alpha_1 d\sigma = 2\pi i a_1,$$

where a_1 is non-negative scalar $m_1 \times m_1$ matrix. Then

$$\int_M \alpha_2 d\sigma = 2\pi i a_2,$$

where a_2 is non-negative scalar $m_2 \times m_2$ matrix such that $\text{trace } a_2 = \text{trace } a_1$. Hence we have

$$\frac{i}{2\pi} \int_M \begin{bmatrix} f_1 - \alpha_1 & 0 \\ 0 & f_2 - \alpha_2 \end{bmatrix} d\sigma = [\mu + a],$$

where $[\]$ denotes the diagonal matrix defined by a vector, so that $[a]$ denotes the matrix $\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$.

But since $a_1 \geq 0, a_2 \leq 0$ with $\text{trace } a_1 = -\text{trace } a_2$ it follows easily that $\mu + a \geq \mu$ with respect to the partial ordering. Hence we have

$$\int_M \Lambda_1(E) d\sigma \geq \frac{i}{2\pi} \int_M \frac{\text{trace}(f_1 - \alpha_1)}{m_1} d\sigma \geq \frac{i}{2\pi} \int_M \frac{\text{trace } f_1}{m_1} d\sigma = \frac{k_1}{m_1}.$$

This completes the proof for the two-step case. The general case proceeds in the same manner and we simply have to keep track of the notation. The details are as follows.

We start with a holomorphic bundle E with its canonical filtration of type μ :

$$0 = E_0 \subset E_1 \subset \dots \subset E_r = E,$$

where the quotients $D_j = E_j / E_{j-1}$ have normalized Chern classes μ^j with

$$\mu^1 > \mu^2 > \dots > \mu^r.$$

The curvature $\Theta(E)$ can then be expressed in a block form generalizing. For every $j < k$ we have an element

$$\eta_{jk} \in \Omega^{0,1}(M, \text{Hom}(D_k, D_j)),$$

so that $d\eta_{jk}$ appears in the (j, k) -block. The η_{jk} are the components of the element

$$\eta_k \in \Omega^{0,1}(M, \text{Hom}(D_k, E_{k-1}))$$

related to the exact sequence

$$0 \rightarrow E_{k-1} \rightarrow E_k \rightarrow D_k \rightarrow 0$$

Now define scalar non-negative $m_j \times m_j$ matrices a_{jk} for $j < k$ by

$$\text{trace } a_{jk} = \frac{1}{2\pi i} \int_M \text{trace}(\eta_{jk} \wedge \eta_{jk}^*) d\sigma \geq 0,$$

and define a_{kk} by

$$\text{trace } a_{kk} = \frac{1}{2\pi i} \int_M \text{trace}(\eta_k^* \wedge \eta_k) d\sigma \leq 0,$$

so that $\sum_{j \leq k} \text{trace } a_{jk} = 0$. Then the convexity leads to the inequality

$$\int_M \Lambda_1(E) d\sigma \geq \frac{i}{2\pi} \int_M \frac{\text{trace}(f_1 - \alpha_1)}{m_1} d\sigma,$$

where a stands for the vector (or diagonal matrix) whose j th block a^j is the scalar (matrix)

$$a^j = \sum_{k \geq j} a_{jk}.$$

Equivalently the vector a can be written as a sum

$$a = \sum b_k,$$

where b_k is the vector corresponding to the diagonal matrix whose j th block is a_{jk} for $j \leq k$ (and zero for $j > k$). The fact that

$$\text{trace } a_{jk} \geq 0 \text{ for } j < l \text{ and } \sum_{j \leq k} \text{trace } a_{jk} = 0$$

implies that $b_k \geq 0$ relative to the partial ordering. Hence $a = \sum b_k \geq 0$ and so $\mu + a \geq \mu$. As before this then implies that

$$\frac{i}{2\pi} \int_M \frac{\text{trace}(f_1 - \alpha_1)}{m_1} d\sigma \geq \frac{i}{2\pi} \int_M \frac{\text{trace } f_1}{m_1} d\sigma = \frac{k_1}{m_1},$$

and so completes the general proof. \square

The theorem can be generalized to Higgs bundles on n -dimension Kähler Manifold [Hu14].

Of course, the theorem implies the following corollary which was also noticed earlier by Möller:

Corollary 4.4. *For a Teichmüller curve C , we have*

$$\varepsilon(C) \geq w(C).$$

Proof. Because

$$\mu_1(f_* \omega_{S/C}) = \frac{\chi}{2},$$

by definition 2.3 and Theorem 4.3, we have

$$\sum_{j=1}^k \varepsilon_i(C) = \int_C \Lambda_j(\omega) d\sigma / \frac{\chi}{2} \geq \sum_{j=1}^k \mu_i(f_* \omega_{S/C}) / \mu_1(f_* \omega_{S/C}) = \sum_{j=1}^k \mu_i(C).$$

\square

4.2. $2\varepsilon \geq \lambda$. The Lyapunov exponents of a vector bundle endowed with a connection also can be viewed as logarithms of mean eigenvalues of monodromy of the vector bundle along a flow on the base.

In the case of the Hodge bundle, we take a fiber of $H_{\mathbb{R}}^1$ and pull it along a Teichmüller geodesic flow on the moduli space. We wait till the geodesic (or Kähler random walks [Ko13]) winds a lot and comes close to the initial point and then compute the resulting monodromy matrix $A(t)$. Finally, let $s_1(t) \geq \dots \geq s_{2g}(t)$ be the eigenvalues of $A^T A$, we compute logarithms of $s_i(t)$ and normalize them by twice the length t of the geodesic

$$\lambda_i = \lim_{t \rightarrow \infty} \frac{\log s_i(t)}{2t}.$$

By the Oseledets multiplicative ergodic theorem, for almost all choices of initial data (starting point, starting direction) the resulting $2g$ real numbers converge as $t \rightarrow \infty$, to limits which do not depend on the initial data within an ergodic component of the flow. These limits $\lambda_1 \geq \dots \geq \lambda_{2g}$ are the Lyapunov exponents of the Hodge bundle along the Teichmüller geodesic flow. ([EKZ11])

A simple corollary of Lemma 4.1 is

Corollary 4.5. *Let $a_1(t) \geq \dots \geq a_{2g}(t)$ be the diagonal elements of $A^T A$, then*

$$\lambda_1 \geq \limsup_{t \rightarrow \infty} \frac{\log a_1(t)}{2t}.$$

We remark here we don't understand any geometry meaning about the right side.

Forni has shown that the eigenvalues of curvature are closely related to Lyapunov exponents. Let $h(c)$ be the unique holomorphic form such that c is the cohomology class of the closed 1-form $\operatorname{Re} h(c)$. By using the first variational formula [FMZ11, p.19]

$$\mathcal{L} \log \|c\|_{\omega} = - \frac{\operatorname{Re} B_{\omega}(h(c), h(c))}{\|c\|_{\omega}^2}.$$

he gets

Corollary 4.6. [Fo02, Corollary 2.2] *Let μ be any $SL(2, \mathbb{R})$ -invariant Borel probability ergodic measure on the moduli space \mathcal{H}_g of normalized Abelian differentials. The second Lyapunov exponent of the Kontsevich-Zorich cocycle with respect to the measure μ , satisfies the following inequality*

$$1 > \int_{\mathcal{H}_g} \sqrt{|\Lambda_2(\omega)|} d\mu(\omega) \geq \lambda_2^{\mu}.$$

Let $\{c_1, \dots, c_k\}$ be any Hodge-orthonormal basis of any isotropic subspace $I_k \subset H_{\mathbb{R}}^1$. His second variational formula [FMZ11, p.21] is

$$\Delta \log \|c_1 \wedge \dots \wedge c_k\|_{\omega} = 2\Phi_k(\omega, I_k)$$

here $w_i = h(c_i)$ and

$$\Phi_k(\omega, I_k) = 2 \sum_{i=1}^k H_{\omega}(\omega_i, \omega_i) - \sum_{i,j=1}^k |B_{\omega}(\omega_i, \omega_j)|^2.$$

Then Theorem 3.1 can be deduced from the following

Corollary 4.7. [Fo02][FMZ11, Corollary 3.2] *Let μ be any $SL(2, \mathbb{R})$ -invariant Borel probability ergodic measure on the moduli space \mathcal{H}_g of normalized Abelian differentials. Assume that there exists $k \in 1, \dots, g-1$ such that $\lambda_k^\mu > \lambda_{k+1}^\mu \geq 0$. Then the following formula holds:*

$$\lambda_1^\mu + \dots + \lambda_k^\mu = \int_{\mathcal{H}_g} \Phi_k(\omega, E_k^+(\omega)) d\mu(\omega).$$

We also give an upper bound of λ_2 :

Corollary 4.8. *Let μ be any $SL(2, \mathbb{R})$ -invariant Borel probability ergodic measure on the moduli space \mathcal{H}_g of normalized Abelian differentials. Then the following formula holds:*

$$2 \int_{\mathcal{H}_g} \Lambda_2(\omega) d\mu(\omega) \geq \lambda_2^\mu.$$

In particular, for a Teichmüller curve

$$2\varepsilon_2 \geq \lambda_2.$$

Proof. For any $c \in \langle [\operatorname{Re}(\omega)], [\operatorname{Im}(\omega)] \rangle^\perp$, Lemma 4.1 implies

$$(5) \quad 2\Lambda_2(\omega) \geq \frac{2H_\omega(h(c), h(c))}{\|c\|_\omega^2} \geq \frac{2H_\omega(h(c), h(c))}{\|c\|_\omega^2} - \frac{|B_\omega(h(c), h(c))|^2}{\|c\|_\omega^4}.$$

Let c be Kontsevich-Zorich cocycle with Lyapunov exponents λ_2^μ . If $\lambda_2^\mu > \lambda_3^\mu$, by Corollary 4.7

$$2 \int_{\mathcal{H}_g} \Lambda_2(\omega) d\mu(\omega) \geq \int_{\mathcal{H}_g} \left(\frac{2H_\omega(h(c), h(c))}{\|c\|_\omega^2} - \frac{|B_\omega(h(c), h(c))|^2}{\|c\|_\omega^4} \right) d\mu(\omega) = \lambda_2^\mu.$$

If $\lambda_2^\mu = \dots = \lambda_k^\mu > \lambda_{k+1}^\mu$, by Corollary 4.7, the result also can be deduced from

$$2(k-1) \int_{\mathcal{H}_g} \Lambda_2(\omega) d\mu(\omega) \geq 2 \int_{\mathcal{H}_g} (\Lambda_2(\omega) + \dots + \Lambda_k(\omega)) d\mu(\omega) \geq \lambda_2^\mu + \dots + \lambda_k^\mu = (k-1)\lambda_2^\mu.$$

□

The fiberwise inequality 5 does not imply $\varepsilon_2 \geq \lambda_2$. Because $\sqrt{|\Lambda_2(\omega)|} \geq 2\Lambda_2(\omega)$ if and only if $\frac{1}{4} \geq \Lambda_2(\omega)$, in contrast to Corollary 4.6, this Corollary is useful when ε_2 is small. Similarly we can get

$$2 \sum_{j=2}^i \varepsilon_j \geq \sum_{j=2}^i \lambda_j \text{ for } i = 2, \dots, n.$$

Or weak form

$$2\varepsilon(C) \geq \lambda(C).$$

(Warning: obviously $2 \sum_{j=1}^g \varepsilon_j \neq 2 \sum_{j=1}^g \lambda_j$. Here \geq only means lying above.)

4.3. Hodge and Newton polygons. Because the conjecture is originally inspired by Katz-Mazur theorem, we intend to introduce some about it.

Let k be a finite field of $q = p^a$ elements; let W denote its ring of Witt vectors, and K the field of fractions of W . Let X be projective and smooth over W , and such that the W -modules $H^r(X, \Omega_{X/W}^s)$ are free (of rank $h^{s,r}$) for all s, r .

Form the polynomial

$$H_m(t) = \prod_{s=0}^m (1 - q^s t)^{h^{s,m-s}} \in Z[t] \subset K[t],$$

which can be called the m -dimensional Hodge polynomial of X/W . set

$$Z_m(t) = \det(1 - F^a |_{H_{DR}^m(X/W)} t) \in K[t],$$

where F is the canonical lifting of Frobenius on de Rham cohomology.

Now, for any polynomial of the form, $R(t) = 1 + R_1 t + R_2 t^2 + \dots + R_\beta t^\beta \in K[t]$. Mazur defined the polygon of $R(t)$ to be the convex closure in the Euclidean plane of the finite set points

$$(j, \text{ord}_q(R_j)), j = 0, 1, \dots, \beta,$$

where $\text{ord}_q(q) = 1$. The left most vertex of this polygon is the origin, while the right-most is $(\beta, \text{ord}_q(R_\beta))$. The structure of this polygon is a measure of the p -adic valuations of the zeros of R .

According to our definition, the convex polygon is

$$(j, \text{ord}_q(R_{\beta-j+1})), j = 0, 1, \dots, \beta.$$

Now what Katz conjectured and Mazur proved is

Theorem 4.9. [Ma72][Ma73] *The convex polygon of $H_m(t)$ (i.e. Hodge polygon) lies above (or on) the convex polygon of $Z_m(t)$ (i.e. Newton polygon).*

A motivation to make the main conjecture is that we try to understand the sentence "Lyapunov exponents as Dynamical Hodge decomposition" [Zo06, p.37].

Corollary 4.10. [Ma72][Ma73] *If the Hodge numbers $h^{s,m-s}$ vanish for $0 \leq s < t$, then the eigenvalues of F^a acting on $H_{DR}^m(X/W)$ are divisible by q^t .*

We think the Proposition 5.14 is an analogy of this Corollary.

5. CONJECTURES

5.1. $\lambda \geq w?$. Zorich asks the following question about the correspondence between Lyapunov exponents and characteristic numbers of some natural bundles. He sets this as the last problem of his survey.

Problem 5.1. [Zo06, p.135] *Study individual Lyapunov exponents of the Teichmüller geodesic flow for all known $SL(2; R)$ -invariant subvarieties; for strata in large genera. Are they related to characteristic numbers of some natural bundles over appropriate compactifications of the strata?*

In fact, originally inspired by Katz-Mazur Theorem 4.9 in crystalline cohomology, we make the following conjecture after checking over all existing numerical examples:

Conjecture 5.2. *For a Teichmüller curve, we have*

$$\lambda(C) \geq w(C).$$

That is

$$\begin{cases} \sum_{j=1}^i \lambda_j \geq \sum_{j=1}^i w_j \text{ for } i = 1, \dots, g-1; \\ \sum_{j=1}^g \lambda_j = \sum_{j=1}^g w_j. \end{cases}.$$

Or equivalently to say : $\sum_{j=i}^g \lambda_j \leq \sum_{j=i}^g w_j$, for $i = 2, \dots, g$; and $\sum_{j=1}^g \lambda_j = \sum_{j=1}^g w_j$.

Remark 5.3. *The result*

$$\varepsilon(C) \geq w(C)$$

in Corollary 4.4 has its root in

$$\Lambda_1(\omega) \geq H_\omega(\omega_i, \omega_i).$$

The result

$$2\varepsilon(C) \geq \lambda(C)$$

in Corollary 4.8 has its root in

$$2\Lambda_1(\omega) \geq \frac{2H_\omega(h(c), h(c))}{\|c\|_\omega^2} - \frac{|B_\omega(h(c), h(c))|^2}{\|c\|_\omega^4}.$$

The Lemma 4.2 or [FMZ11, p.22] gives us

$$\frac{2H_\omega(h(c), h(c))}{\|c\|_\omega^2} - \frac{|B_\omega(h(c), h(c))|^2}{\|c\|_\omega^4} \geq \frac{H_\omega(h(c), h(c))}{\|c\|_\omega^2}.$$

Maybe the conjecture will have it root in this.

When the equality is reached, we also make the following 'rigid' conjecture:

Conjecture 5.4. *If*

$$\sum_{j=1}^k \lambda_j = \sum_{j=1}^k w_j$$

and

$$w_k \neq w_{k+1}$$

then VHS contain a rank $2(g-k)$ Local system.

It can be considered as the inverse to Kontsevich-Forni's formula 3.1. We will show a simple corollary of the two conjecture in Proposition 5.14.

Problem 5.5 (Continuous assumption). *We also hope one can define g -vector $w(M)$ for any Teichmüller geodesic flows M , which should satisfy the following natural continuous properties:*

- (1) *If $w_i(C)$ are constants for any Teichmüller curves C in a Teichmüller geodesic flows M , then $w_i(M) = w_i(C)$;*
- (2) *If $w_i(C) \leq a$ for any Teichmüller curves C in a Teichmüller geodesic flows M , then $w_i(M) \leq a$.*

In particular, maybe the following Eskin-Mirzakhani-Mohammadi theorem is useful for generalizing it to Teichmüller geodesic flows:

Theorem 5.6. [EMM13, Theorem 2.3] *Let \mathcal{N}_n be a sequence of affine manifolds, and suppose $\nu_{\mathcal{N}_n} \rightarrow \nu$. Then ν is a probability measure. Furthermore, ν is the affine measure $\nu_{\mathcal{N}}$, where \mathcal{N} is the smallest submanifold with the following property: there exists some $n_0 \in \mathbb{N}$ such that $\mathcal{N}_n \subset \mathcal{N}$ for all $n > n_0$.*

Eskin, Bonatti and Wilkinson use this theorem and a theorem of Filip [Fi13a], answers the following question asked by Matheus, Möller and Yoccoz in [MMY13].

Theorem 5.7. [EBW14, Theorem 5] *Let \mathcal{N}_n be a sequence of affine manifolds, and suppose $\nu_{\mathcal{N}_n} \rightarrow \nu$. Then the Lyapunov exponents of $\nu_{\mathcal{N}_n}$ converge to the Lyapunov exponents of ν .*

Remark 5.8. *Teichmüller curves, in fact already those that are generated by square-tiled surfaces, are dense in each stratum [Mö12, p.14]. For each stratum, there is a sequence of Teichmüller curves converge to it in the sense of Theorem 5.6. By Theorem 5.7, the sequence of the Lyapunov spectrum of those Teichmüller curves also converge to the Lyapunov spectrum of the Teichmüller geodesic flow.*

Sometime we use the convergent properties of the Lyapunov spectrum to replace the continuous assumption 5.5 of the Harder-Narasimhan spectrum.

In the Appendix, we list approximate values of individual Lyapunov exponents λ_i of the connected components in small genera. We also list our w_i for all Teichmüller curves in these connected component. The reader can check the numerical evidences by himself. Here we check a simple case below.

Example 5.9. *In the stratum $\mathcal{H}_5^{\text{odd}}(6, 2)$, First by continuity assumption 5.5, the w_i for the components are the same as that for all Teichmüller curves. Now from Table 6, 6, we have*

$$\begin{aligned} \lambda_1 &= 1 &= 1 &= w_1, \\ \lambda_1 + \lambda_2 &\approx 1.52 &\geq 10/7 &= w_1 + w_2, \\ \lambda_1 + \lambda_2 + \lambda_3 &\approx 1.89 &\geq 37/21 &= w_1 + w_2 + w_3, \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 &\approx 2.1 &\geq 43/21 &= w_1 + w_2 + w_3 + w_4, \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 &= 46/21 &= 46/21 &= w_1 + w_2 + w_3 + w_4 + w_5. \end{aligned}$$

Or equivalently to say

$$\begin{aligned} \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 &= 46/21 &= 46/21 &= w_1 + w_2 + w_3 + w_4 + w_5, \\ \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 &\approx 25/21 &= 25/21 &= w_2 + w_3 + w_4 + w_5, \\ \lambda_3 + \lambda_4 + \lambda_5 &\approx 0.67 &\leq 16/21 &= w_3 + w_4 + w_5, \\ \lambda_4 + \lambda_5 &\approx 0.30 &\leq 3/7 &= w_4 + w_5, \\ \lambda_5 &\approx 0.09 &\leq 1/7 &= w_5. \end{aligned}$$

Of course we can use Remark 5.8 to replace the continuity assumption 5.5 in the discussion.

Remark 5.10 (Finiteness of the algebraically primitive). *The following discussion is an analogy to the proof in [EBW14]. For an algebraically primitive Teichmüller curve, we have w_i equals λ_i [YZ12b]. If there exist infinitely many algebraically primitive Teichmüller curves \mathcal{N}_n in the stratum $\mathcal{H}_3(1, 1, 1, 1)$. By Table 1, we have*

$$\frac{1}{2} \geq w_2(\mathcal{N}_n) = \lambda_2(\mathcal{N}_n)$$

By Theorem 5.6, the \mathcal{N}_n have to converge to an affine manifold \mathcal{N} (in the sense that the affine measures $\nu_{\mathcal{N}_n}$ will converge to the affine measure $\nu_{\mathcal{N}}$). By Theorem 5.7,

$$\frac{1}{2} \geq \lambda_2(\mathcal{N}).$$

But the approximate values of $\lambda_2(\mathcal{H}_3(1, 1, 1, 1))$ for the stratum $\mathcal{H}_3(1, 1, 1, 1)$ is

$$\lambda_2(\mathcal{H}_3(1, 1, 1, 1)) \approx 0.5517 > \frac{1}{2},$$

so the algebraically primitive Teichmüller curves are not dense in $\mathcal{H}_3(1, 1, 1, 1)$. The method can be used for any Teichmüller geodesic flow which one can get approximate values of λ_i .

In fact, Matheus and Wright have shown that algebraically primitive Teichmüller curves are not dense in any connected component of any stratum in genus at least 3 [MW13].

Remark 5.11 (Main conjecture is correct for almost all Teichmüller curves in low genus strata). *The method of Remark 5.10 and the computation of Example 5.9 also show that the Teichmüller curves which doesn't satisfy the conjecture 5.2 are not dense in those strata which we list in the Table 6, 6.*

Remark 5.12 (Simplicity of the Lyapunov spectrum). *In the stratum $\mathcal{H}_3^{hyp}(4)$, all Teichmüller curves have $w_1 = 1, w_2 = 3/5, w_3 = 1/5$, so according to Conjecture 5.2, we have*

$$\lambda_2 \geq 3/5 = w_2 > w_3 = 1/5 \geq \lambda_3,$$

the Lyapunov spectrum is simple. It has been proved for all but finite Teichmüller curves by Matheus, Möller and Yoccoz [MMY13], reproved in [EBW14].

By Theorem 3.3, the w_i of all Teichmüller curves in $\overline{\mathcal{H}}_g^{hyp}(2g-2)$ and $\overline{\mathcal{H}}_g^{hyp}(g-1, g-1)$ are

$$WSpec = \begin{cases} \frac{1}{N}, \frac{3}{N}, \frac{5}{N}, \dots, \frac{N}{N} & N = 2g-1 \\ \frac{2}{N}, \frac{4}{N}, \frac{6}{N}, \dots, \frac{N}{N} & N = 2g \end{cases}$$

If our Conjecture 5.2 is true, then

$$\lambda_2 \geq \frac{N-2}{N},$$

where $N = 2g-1$ in case of $\overline{\mathcal{H}}_g^{hyp}(2g-2)$ and $N = 2g$ in case of $\overline{\mathcal{H}}_g^{hyp}(g-1, g-1)$. Since

$$1 \geq \lim_{g \rightarrow \infty} \lambda_2 \geq \lim_{N \rightarrow \infty} \frac{N-2}{N} = 1.$$

The following conjecture is a corollary of Conjecture 5.2 and Remark 5.8 or continuous assumption 5.5:

Conjecture 5.13 (Kontsevich-Zorich [KZ97]). *For the hyperelliptic components $\overline{\mathcal{H}}_g^{hyp}(2g-2)$ and $\overline{\mathcal{H}}_g^{hyp}(g-1, g-1)$,*

$$\lim_{g \rightarrow \infty} \lambda_2 = 1.$$

5.2. Higgs fields. For the weight one \mathbb{Q} -VHS $(\mathbb{V}, H^{1,0} = f_*\omega_{S/C} \subset H = (\mathbb{V} \otimes_{\mathbb{Q}} \mathcal{O}_{(C \setminus \Delta)})_{ext})$ which comes from the semi-stable family of curves $f: S \rightarrow C$. The connection ∇ composed with the inclusion and projection give a map

$$\theta^{1,0}: H^{1,0} \rightarrow H \rightarrow H \otimes \Omega_C(\log \Delta) \rightarrow (H/H^{1,0}) \otimes \Omega_C(\log \Delta),$$

which is \mathcal{O}_C -linear. If we extend $\theta^{1,0}$ by zero mapping to the associated graded sheaf we get a Higgs bundle $(\text{gr}(H), \theta) = (H^{1,0} \oplus H^{0,1}, \theta^{1,0} \oplus 0)$. By definition this is a vector bundle on C with a holomorphic map $\theta: F \rightarrow F \otimes \Omega_C(\log \Delta)$, the additional $\theta \wedge \theta$ being void if the base is a curve. ([VZ04],[Mö06])

Sub-Higgs bundles of a Higgs bundle (F, θ) are subbundles $G \subset F$, such that $\theta(G) \subset G$. The Higgs bundle is stable if for any sub-Higgs bundle $(G, \theta|_G)$

$$\frac{\deg(G)}{\text{rk}(G)} < \frac{\deg(F)}{\text{rk}(F)}.$$

Semi-stable is defined similarly but we allow now the weak inequality $\deg(G)/\text{rk}(G) \leq \deg(F)/\text{rk}(F)$.

A Higgs bundle (F, θ) is polystable if

$$(F, \theta) = \bigoplus_i (F_i, \theta_i) \text{ where } (F_i, \theta_i) \text{ are stable Higgs bundles.}$$

Simpson shows that every stable Higgs bundle (F, θ) has a Hermitian-Yang-Mills metric. If $c_1(F) = 0$, $\theta \wedge \theta = 0$ and $c_2(F)[\omega]^{n-2} = 0$ then the connection is flat. the last two condition is void if the base is a curve. Simpson also shows that for the complex variation of the Hodge structure H , $(\text{gr}(H), \theta)$ is a polystable Higgs bundle such that each direct summand is of degree 0. ([Si88])

Simpson's correspondence allows us to switch back and forth between degree 0 stable sub-Higgs bundles of F and sub-local systems of \mathbb{V} .

The Higgs field $(\text{gr}(H), \theta)$ is the edge morphism

$$H^{1,0} = f_*\omega_{S/C} \rightarrow R^1 f_* \mathcal{O}_S \otimes \Omega_C^1(\log \Delta) = H^{0,1} \otimes \Omega_C(\log \Delta)$$

of the tautological sequence

$$0 \rightarrow f^* \Omega_C^1(\log \Delta) \rightarrow \Omega_S^1(\log(f^{-1}\Delta)) \rightarrow \Omega_{S/C}^1(\log(f^{-1}\Delta)) \rightarrow 0$$

The results about Higgs bundles are well known for experts. ([Ko87],[VZ04])

Proposition 5.14. *For a Teichmüller curve, we have*

$$w_i = 0 \Rightarrow \lambda_i = 0.$$

If $w_k \neq 0, w_{k+1} = \dots = w_g = 0$, then VHS contains a sub-local system of \mathbb{V} of rank $2(n-k)$.

Proof. For any stable quotient bundle

$$H^{1,0} = f_*\omega_{S/C} \rightarrow V \rightarrow 0,$$

by dualization we have

$$0 \rightarrow V^\vee \rightarrow f_*\omega_{S/C}^\vee = H^{0,1}.$$

Then we construct a stable sub-Higgs bundle:

$$(0 \oplus V^\vee, 0) \subset (H^{1,0} \oplus H^{0,1}, \theta^{1,0} \oplus 0).$$

Since $(\text{gr}(H), \theta)$ is a polystable Higgs bundle, then this means that

$$-\deg(V) = \deg(0 \oplus V^\vee) \leq \deg(H^{1,0} \oplus H^{0,1}) = 0.$$

Denote by $HN_{min}(W)$ the last quotient in the Harder-Narasimhan filtration of a vector bundle W . Let $\mu_{min}(W)$ be the slope $\mu(HN_{min}(W))$. Then

$$\mu_{min}(f_*\omega_{S/C}) \geq 0.$$

Because any quotient bundle $f_*\omega_{S/C} \xrightarrow{\phi} Q \rightarrow 0$ induces a quotient bundle $f_*\omega_{S/C} \xrightarrow{\phi} HN_{min}(Q) \rightarrow 0$, we have $\mu_{min}(Q) \geq \mu_{min}(f_*\omega_{S/C})$ (otherwise the map ϕ is zero). We then obtain

$$\deg(Q) \geq \mu_{min}(Q) \cdot \text{rk}(Q) \geq \mu_{min}(f_*\omega_{S/C}) \cdot \text{rk}(Q) \geq 0.$$

If $w_k \neq 0$ and $w_{k+1} = \dots = w_g = 0$, then $\mu_{min}(f_*\omega_{S/C}) = 0$. Since $(\text{gr}(H), \theta)$ is a polystable Higgs bundle, then this means that

$$(0 \oplus HN_{min}^\vee(f_*\omega_{S/C}), 0) \text{ is a direct summand of } (\text{gr}(H), \theta).$$

So $HN_{min}^\vee(f_*\omega_{S/C})$ is a direct summand of $H^{0,1}$ and $HN_{min}(f_*\omega_{S/C})$ is a direct summand of $f_*\omega_{S/C}$. Furthermore [VZ04, p.1]

$$(HN_{min}(f_*\omega_{S/C}) \oplus HN_{min}^\vee(f_*\omega_{S/C}), 0)$$

is a polystable Higgs bundle such that each direct summand has degree 0, so it comes from a sub-local system of \mathbb{V} of rank $2(n-k)$.

By Theorem 3.1, we have:

$$0 \leq \lambda_{k+1} + \dots + \lambda_g \leq 2\deg(HN_{min}(f_*\omega_{S/C}))/ (2g(C) - 2 + |\Delta|) = 0$$

$$\implies \lambda_{k+1} = \dots = \lambda_g = 0.$$

□

The reader can compare this Proposition with [FMZ11, Theorem 3]. It can be used to get some information of the zero eigenvalues of $EV(H_\omega)$ and $EV(B_\omega)$.

5.3. Quadratic differentials. For a Teichmüller curve C generated by (Y, q) in $\mathcal{Q}(d_1, \dots, d_s)$, let (X, ω) be the canonical double covering. The curve X comes with an involution τ . Its cohomology splits into the τ -invariant and τ -anti-invariant part. Adapting the notation of [EKZ11] we let $g = g(Y)$ and $g_{eff} = g(X) - g$. Let λ_i^+ be the Lyapunov exponents of the τ -invariant part of $H^1(X, \mathbb{R})$ and λ_i^- be the Lyapunov exponents of the τ -anti-invariant part. The τ -invariant part descends to Y and hence the λ_i^+ are the Lyapunov exponents of (Y, q) we are primarily interested in. Define

$$L^+ = \lambda_1^+ + \dots + \lambda_g^+$$

$$L^- = \lambda_1^- + \dots + \lambda_{eff}^-$$

The role of λ_i^+ is analogous to the ordinary sum of Lyapunov exponents in the case of abelian differentials. We will reprove the following formula in [KZ97, p.12][EKZ11, p.12] by using double cover techniques [BHPV03, p.236]. It is largely based on the work of Chen-Möller [CM12].

Proposition 5.15. *For a Teichmüller curve C in $\mathcal{Q}(d_1, \dots, d_s)$, we have*

$$L^- - L^+ = \frac{1}{4} \sum_{\substack{j \text{ such that} \\ d_j \text{ is odd}}} \frac{1}{d_j + 2}.$$

Proof. Note that

$$\phi: \mathcal{Q}(\dots, d_i, \dots, d_j, \dots) \rightarrow \mathcal{H}_g(\dots, d_i/2, d_i/2, \dots, d_j + 1, \dots)$$

for d_i even and for d_j odd. Since the double cover is branched at the singularities of odd order. Restrict this to a Teichmüller curve C in $\mathcal{Q}(d_1, \dots, d_n)$. Then it gives rise to a Teichmüller curve isomorphic to C in the corresponding stratum of abelian differentials. After suitable base change and compactification, we can get two universal families f', f and we have the following commutative diagram

$$\begin{array}{ccc} S' & \xrightarrow{\sigma} & S \\ & \searrow f' \quad \swarrow f & \\ & C & \end{array}$$

and let D'_j be the section of $f': S' \rightarrow C$ over D_j in case d_j is odd and $D_{j,1}, D_{j,2}$ be the sections over D_j in case d_j is even.

In the case when d_j is odd, we have [CM12, p.14]

$$\sigma_* D'_j = D_j, \sigma^* D_j = 2D'_j,$$

and the self-intersection number is

$$D_j^2 = (\sigma_* D'_j) D_j = 2(D_j'^2) = -\frac{\chi}{d_j + 2}.$$

In the case when d_j is even, we have

$$\sigma_*(D_{j,1} + D_{j,2}) = 2D_j, \sigma^* D_j = D_{j,1} + D_{j,2},$$

and the self-intersection number is

$$D_j^2 = \frac{1}{2}(\sigma_*(D_{j,1} + D_{j,2})) D_j = \frac{1}{2}(D_{j,1}^2 + D_{j,2}^2) = -\frac{\chi}{d_j + 2}.$$

The relative canonical bundle formula for the fibration $f': S' \rightarrow C$ is

$$\omega_{S'/C} = f'^* \mathcal{L} \otimes \mathcal{O}_{S'} \left(\sum_{\substack{j \text{ such that} \\ d_j \text{ is even}}} \frac{d_j}{2} (D_{j,1} + D_{j,2}) + \sum_{\substack{j \text{ such that} \\ d_j \text{ is odd}}} (d_j + 1) D'_j \right),$$

where \mathcal{L} is the line bundle on C corresponding to the generating abelian differential and $\deg \mathcal{L}$ equals $\chi/2$.

The relative canonical bundle formula for the fibration $f: S \rightarrow C$ is

$$\omega_{S/C}^2 = f^* \mathcal{F} \otimes \mathcal{O}_S \left(\sum d_j D_j \right),$$

where \mathcal{F} is the line bundle on C corresponding to the generating quadratic differential and obviously \mathcal{F} equals \mathcal{L}^2 . ([CM12, p.42])

There is a smooth divisor

$$B = \sum_{\substack{j \text{ such that} \\ d_j \text{ is odd}}} D_j$$

on S , such that $B = 2D$ for some effective divisor D . The double covering $\sigma: S' \rightarrow S$ is ramified exactly over B . We have

$$\omega_{S'} = \sigma^*(\omega_S \otimes \mathcal{O}_S(D))$$

and

$$\sigma_*(\mathcal{O}_{S'}) = \mathcal{O}_S \oplus \mathcal{O}_S(-D).$$

The direct image of relative canonical bundle $f'_*\omega_{S'/C}$ decomposes into a direct sum

$$\begin{aligned} f'_*\omega_{S'/C} &= f'_*\omega_{S'} \otimes \omega_C^{-1} = f_*\sigma_*\omega_{S'} \otimes \omega_C^{-1} \\ &= f_*(\sigma_*(\mathcal{O}_{S'}) \otimes \omega_S \otimes \mathcal{O}_S(D)) \otimes \omega_C^{-1} \\ &= f_*(\omega_S \otimes \mathcal{O}_S(D) \oplus \omega_S) \otimes \omega_C^{-1} \\ &= f_*(\omega_{S/C} \oplus f_*(\omega_{S/C} + D)). \end{aligned}$$

We assume $D \neq 0$ (when $D = 0$, the result is trivial.). Then all the higher direct images of $\omega_{S/C} + D$ are zero. By Grothendieck-Riemann-Roch we have

$$\begin{aligned} ch(f_*(\omega_{S/C} + D)) &= f_*(ch(\omega_{S/C}) \cdot ch(D) \cdot (1 - \frac{\gamma}{2} + \frac{\gamma^2 + \eta}{12})) \\ &= f_*((1 + \gamma + \frac{\gamma^2}{2}) \cdot (1 + D + \frac{D^2}{2}) \cdot (1 - \frac{\gamma}{2} + \frac{\gamma^2 + \eta}{12})) \\ &= f_*(1 + (\frac{\gamma}{2} + D) + \frac{D(D + \gamma)}{2} + \frac{\gamma^2 + \eta}{12}) \\ &= \text{rank} + (f_*\frac{D(D + \gamma)}{2} + \lambda), \end{aligned}$$

where $\gamma = c_1(\omega_{S/C})$, $\lambda = c_1(f_*\omega_{S/C})$ and η the nodal locus in $f: S \rightarrow C$, they satisfy $\lambda = \frac{\gamma^2 + \eta}{12}$ by Riemann-Roch. Now we have

$$c_1(f_*(\omega_{S/C} + D)) - c_1(f_*\omega_{S/C}) = c_1(f_*(\omega_{S/C} + D)) - \lambda = f_*\frac{D(D + \gamma)}{2}.$$

Because

$$\begin{aligned} \deg(f_*(\omega_{S/C} + D)) - \deg(f_*\omega_{S/C}) &= \frac{1}{2}D(\omega_{S/C} + D) \\ &= \frac{1}{8}(\sum_{\substack{j \text{ such that} \\ d_j \text{ is odd}}} D_j)(\sum_{\substack{j \text{ such that} \\ d_j \text{ is odd}}} D_j + \sum d_j D_j + \mathcal{F}) \\ &= \frac{1}{8}(\sum_{\substack{j \text{ such that} \\ d_j \text{ is odd}}} (d_j + 1)D_j^2 + \chi) \\ &= \frac{\chi}{2} \frac{1}{4} \sum_{\substack{j \text{ such that} \\ d_j \text{ is odd}}} \frac{1}{d_j + 2}, \end{aligned}$$

and $f_*(\omega_{S/C})$ is σ -invariant part and $f_*(\omega_{S/C} + D)$ is σ -anti-invariant part. Hence we have

$$L^- = \deg(f_*(\omega_{S/C} + D))/\frac{\chi}{2}, L^+ = \deg(f_*\omega_{S/C})/\frac{\chi}{2}.$$

We get the formula

$$L^- - L^+ = \frac{1}{4} \sum_{\substack{j \text{ such that} \\ d_j \text{ is odd}}} \frac{1}{d_i + 2}.$$

□

Remark 5.16. Remark 5.8 or continuous assumption 5.5 can be used to generalize the Proposition to each stratum.

For the Teichmüller curve C in $\overline{\mathcal{H}}_g(\dots, d_i/2, d_i/2, \dots, d_j + 1, \dots)$, we have defined $g + g_{eff}$ -vector

$$w(C) = (w_1, \dots, w_{g+g_{eff}}).$$

and moreover we know the upper bound of each w_i . Because

$$\text{grad}(HN(f'_* \omega_{S'/C})) = \text{grad}(HN(f_*(\omega_{S/C}))) \oplus \text{grad}(HN(f_*(\omega_{S/C} + D))),$$

we can divide w_i into two parts

$$w_1^+, \dots, w_g^+; w_1^-, \dots, w_{eff}^-.$$

Here w_1^+, \dots, w_g^+ (resp. w_1^-, \dots, w_{eff}^-) come from the graded quotient $\text{grad}(HN(f_*(\omega_{S/C})))$ (resp. $\text{grad}(HN(f_*(\omega_{S/C} + D)))$).

It is obvious by definition

$$L^+ = w_1^+ + \dots + w_g^+, L^- = w_1^- + \dots + w_{eff}^-.$$

Example 5.17. Consider the map

$$\phi: \mathcal{Q}(1, 2, -1, -1, -1) \rightarrow \mathcal{H}_2(2, 1, 1).$$

For a Teichmüller curve in $\mathcal{H}_2(2, 1, 1)$, we know that

$$w_1 = 1, w_2 = 1/2, w_3 = 1/3.$$

For a Teichmüller curve in $\mathcal{Q}(1, 2, -1, -1, -1)$, we have

$$L^- - L^+ = 5/6 = 1/4(1/3 + 1 + 1 + 1), L^- + L^+ = w_1 + w_2 + w_3 = 11/6.$$

We get $L^- = 4/3, L^+ = 1/2$, and so give us

$$w_1^+ = 1/2, w_1^- = 1, w_2^- = 1/3.$$

Of course we can ask the same questions for $w_i^-(w_i^+)$ and $\lambda_i^-(\lambda_i^+)$.

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6. APPENDIX

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TABLE 2. w_i for all Teichmüller curves in genus 4. (Theorem 3.3 and 3.4)

zeros	component	w_2	w_3	w_4	$\sum w_i$
(6)	hyp	5/7	3/7	1/7	16/7
(6)	even	4/7	2/7	1/7	14/7
(6)	odd	3/7	2/7	1/7	13/7
(5,1)		1/2	2/6	1/6	2
(3,3)	hyp	3/4	2/4	1/4	5/2
(3,3)	non-hyp	2/4	1/4	1/4	2
(4,2)	even	3/5	1/3	1/5	32/15
(4,2)	odd	2/5	1/3	1/5	29/15
(2,2,2)	odd	1/3	1/3	1/3	2
(3,2,1)		1/2	1/3	1/4	25/12
(2,2,2)	even	$\leq 2/3$	$\leq 1/3$	$\leq 1/3$	$\leq 7/3$
(1,1,1,3)		$\leq 1/2$	$\leq 1/2$	$\leq 1/4$	$\leq 9/4$
(1,1,2,2)		$\leq 2/3$	$\leq 1/2$	$\leq 1/3$	$\leq 5/2$
(1,1,1,1,2)		$\leq 1/2$	$\leq 1/2$	$\leq 1/3$	$\leq 7/3$
(1,1,1,1,1,1)		$\leq 1/2$	$\leq 1/2$	$\leq 1/2$	$\leq 5/2$

TABLE 3. w_i for all Teichmüller curves in genus 5. (Theorem 3.3 and 3.4)

zeros	component	w_2	w_3	w_4	w_5	$\sum w_i$
(8)	hyp	7/9	5/9	3/9	1/9	25/9
(8)	even	5/9	3/9	2/9	1/9	20/9
(8)	odd	4/9	3/9	2/9	1/9	19/9
(5,3)		1/2	1/3	1/4	1/6	9/4
(6,2)	odd	3/7	1/3	2/7	1/7	46/21
(4,4)	hyp	4/5	3/5	2/5	1/5	3
(7,1)		$\leq 3/4$	$\leq 1/2$	$\leq 3/8$	$\leq 1/8$	$\leq 11/4$
(6,2)	even	$\leq 5/7$	$\leq 4/7$	$\leq 1/3$	$\leq 1/7$	$\leq 58/21$
(6,1,1)		$\leq 5/7$	$\leq 1/2$	$\leq 3/7$	$\leq 1/7$	$\leq 39/14$
(5,2,1)		$\leq 2/3$	$\leq 1/2$	$\leq 1/3$	$\leq 1/6$	$\leq 8/3$
(5,1,1,1)		$\leq 2/3$	$\leq 1/2$	$\leq 1/2$	$\leq 1/6$	$\leq 17/6$

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TABLE 4. Approximate values of individual Lyapunov exponents in small genera: genus 4. ([KZ97],[EKZ11])

zeros	component	λ_2	λ_3	λ_4	$\sum \lambda_i$
(6)	hyp	0.7375	0.4284	0.1198	16/7
(6)	even	0.5965	0.2924	0.1107	14/7
(6)	odd	0.4733	0.2755	0.1084	13/7
(5,1)		0.5459	0.3246	0.1297	2
(3,3)	hyp	0.7726	0.5182	0.2097	5/2
(3,3)	non-hyp	0.5380	0.3124	0.1500	2
(4,2)	even	0.6310	0.3496	0.1527	32/15
(4,2)	odd	0.4789	0.3134	0.1412	29/15
(2,2,2)	odd	0.4826	0.3423	0.1749	2
(3,2,1)		0.5558	0.3557	0.1718	25/12
(2,2,2)	even	0.6420	0.3785	0.1928	737/333
(1,1,1,3)		0.5600	0.3843	0.1849	66/31
(1,1,2,2)		0.5604	0.3809	0.1982	5045/2358
(1,1,1,1,2)		0.5632	0.4032	0.2168	131/60
(1,1,1,1,1,1)		0.5652	0.4198	0.2403	839/377

TABLE 5. Approximate values of individual Lyapunov exponents in small genera: genus 5. ([KZ97],[EKZ11])

zeros	component	λ_2	λ_3	λ_4	λ_5	$\sum \lambda_i$
(8)	hyp	0.798774	0.586441	0.305803	0.086761	25/9
(8)	even	0.597167	0.362944	0.189205	0.072900	20/9
(8)	odd	0.515258	0.343220	0.181402	0.071101	19/9
(5,3)		0.561989	0.376073	0.216214	0.095789	9/4
(6,2)	odd	0.521181	0.368690	0.211988	0.088735	46/21
(7,1)		0.560205	0.378184	0.206919	0.081789	2423/1088
(6,2)	even	0.603895	0.385796	0.220548	0.091624	178429/77511
(6,1,1)		0.563306	0.398655	0.229768	0.093637	59332837/25961866
(5,2,1)		0.564138	0.396293	0.236968	0.103124	4493/1953
(5,1,1,1)		0.565422	0.414702	0.252838	0.107906	103/44

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